**Thermal Properties**

**Ginzburg-Landau Free Energy (Functional)**

We’d like to further investigate the thermodynamics of a superconductor. We constructed the Free Energy of our superconductor from the BCS model. But we will want to generalize our discussion to include the possibility of an external electromagnetic field. We can also do this with path integrals. Let’s go back to our H.



Actually going to use the position space δ function interaction approximation instead,



Referring to the Stat Mech/G, Z as Path Integral file, and QM/Time-Dependent/Many Identical Particles/Propagator as Path Integral file, we have:



where,



and K = H – μN. Filling in our particular H, we get:



Now let’s add in an EM field. We’ll recall this is done by making transformation p → p – eA, i.e., -i∇ → -i∇ - eA, where A is the magnetic vector potential. We’ll let A be position -dependent. So,



As is standard, we will want to use a Hubbard-Stratonovich identity thing to turn the quartic term into a quadratic term,



(recall from Path Integrals file the identity,



and the det(**A**) part is just an irrelevant constant so we’re ignoring it) And we can move the ’s and *ψ*’s around at cost of minus signs of course. Filling this in (and leaving the arguments off the ψ’s and Δ’s, except in the measure, to save space),



We can do the ψ integration, but only perturbatively, unless we pretend A and Δ are constants. So we’ll want to separate out the free part of S from the A’s and Δ’s. We’ll also want to put everything in matrix form. To that end, let’s introduce, as we have before in previous files, the Nambu spinors.



In terms of this, we can write the action as:



The bottom right element of the matrix requires IBP and some anticommutation of the ψ’s to put in the given order. Going to make a few definitions before we proceed. First,



Now let’s separate out the A’s:



So we can write,



where **G**0-1 = ([G11|0-1, 0], [0, G22|0-1, 0]), **σ**z = ([1,0], [0,-1]) is the Pauli spin matrix. So altogether,



Alright. We can write our Ξ as,



Now we’re in a position to perturbatively do the ψ integration. We’ll write,



The Δ\*Δ term we can ignore for now, since it isn’t coupled to the ψ’s. And to prep the ψ integration, we will separate out the free part of Sψ from the interacting part of Sψ (the part with the A’s and Δ’s):



And then our total result would be:



where of course,



Now we have to see how we can diagrammatically calculate this < >free.

**Feynman rules**

Okay well we need the Feynman rules. To that end I’m going to simplify the action a little. Remember that first grad acts on everything to its right…but we can IBP to make it act on the instead,



where,



Then the Feynman rules are:

Diagram, schematic

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where the GF is given by:



Note the grads on first diagram act in different directions. The 1st one acts on the second argument of the GF to A’s left, and the 2nd one acts on the first argument of the GF to A’s right. It’s better, though, to convert to Fourier space rules. See the FT file for ideas, and also might check out the Cond Mat/Metals/Electrons and Impurities/Conductivity/Quantum file for another instance where the leg gets a momentum factor.

Diagram

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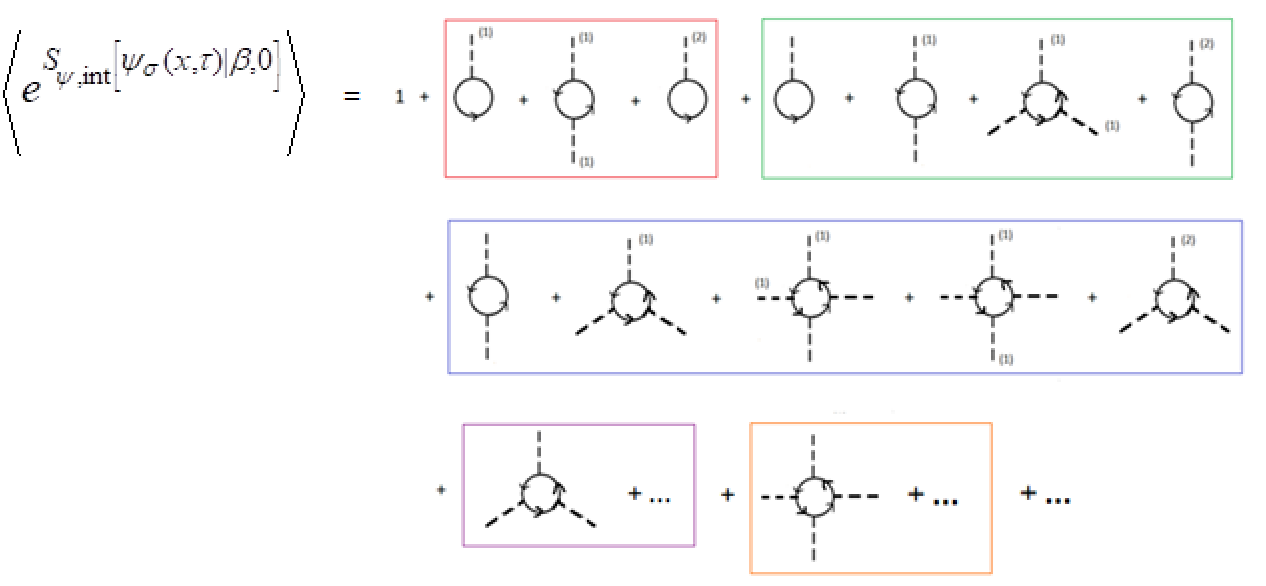
where the GF is:



Note the A2(q,iνm) is to be interpreted as the spatio-temperal FT of A2(x,τ) – not obvious from the notation, granted. The frequency index on the A’s is bosonic, because it will be the difference between an incoming and outgoing fermion leg, and the difference between two fermion/odd frequencies is a bosonic/even frequency. And for our case, we’ll say A is actually time-independent. As usual we conserve energy/frequency and momentum/wavenumber at every vertex, and sum over all frequencies and wavenumbers, with additional factor of 1/β and 1/V for each k and iωn d.o.f. And we are summing over indices too, and since all diagrams form a closed loop, this reduces to a Trace over the product of matrices in the diagram. And we have the customary (-1) for every Fermion loop. Anyway, so we’ll recall our Free Energy is given by:



and up to second order in A, and second order in Δ, this will correspond to the following diagrammatic expansion (left off the free),



So many terms ☹. Actually went a little past 2nd order. The red box encloses the O(Δ0) terms, the green box the O(Δ1) terms, the blue box the O(Δ2) terms, the purple box the O(Δ3) terms, and the orange box the O(Δ4) terms, etc.

**Doing the Calculation**

So now we’ll go mostly term by term through the expansion.

**O(Δ0) terms**

I won’t bother calculating these. We calculated them in the Cond Matt/Free Day/Electrons/Thermal Properties/Magnetic Susceptibility (Path Integral) file. And these, together with Ξfree constitute the free energy of a free electron gas in a magnetic field.

**O(Δ1) terms**

So the first guy is:

Diagram

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we get:



And in fact we can see that all O(Δ1) terms will be zero, because they involve traces of diagonal matrices with a single off-diagonal matrix, **Δ**, and so the Tr will be 0.

**O(Δ2) terms**

Okay, the first term is:

Diagram

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And so we have,



where in the last couple lines we change summation variables in the second GF term: k→k-q, iωn→iωn-iνm, and then q→-q, iνm → -iνm. Now note that Δ\*(q,iνm) means the Fourier transform of Δ\*(x,τ). And that Δ\*(q,iνm) = Δ(-q,-iνm)\*, where Δ(q,iνm)\* is the complex conjugate of the Fourier transform of Δ(x,τ). since,



So this works out to,



where we define the Cooperon polarizion bubble.



We can do the frequency summation,



(‘cause nF is periodic in bosonic frequencies) and well,



So,



We already encountered this guy in the Excitations folder/pair-pair correlation file. We called it Λ. We’re going to make the semi-classical approximation here. This amounts to saying that Δ is independent of (complex) time. If so, then the Fourier transform of Δ w/r to τ just gives a β×delta function, and eliminates the summation over νn. So we’re going to say,



Going to do a small q expansion of the Cooper pair thing. First going to shift variables.



Keeping only to O(q2),



So we can say,



where,



It would appear that the Π0(2a) term diverges with large k, but keep in mind that there really should be a momentum cutoff to within ωD of the Fermi surface. This would’ve been evident had we started with the Hamiltonian with that explicit cut-off. But we started with a delta function model instead, which has no cutoff. So that’s why our integral diverges. There also seems to be a divergence at ξκ = 0, but that’s okay because it diverges linearly, while the measure goes to zero quadratically. In that Excitations folder/Pair-Pair correlation function file, we found the zeroth order term was,



where ρF is the density of states at the Fermi surface. Can look in that file, but it goes to 1/λ at the critical temperature Tc. So close to the critical point, we could write, defining δT = T – Tc,



The second guy is:



where we use the isotropy of the **k**·**q** term to simplify in the second line. I don’t want to work this term out. But clearly its units are density of states × energy. Evidently, we would find,



where ρF\* is some ‘renormalized’ density of states at the Fermi surface – really, just ρF times some prefactor. It looks like the prefactor will be ½. So we’ll say,



Going back to position space, this will be…could just fill in Δ(q) = FTq[Δ(q)] = ∫d3x eiq·xΔ(x) and work it all out, but going to try to do this more quickly, using transform rules – see FT file again – well okay first note/verify that:



and therefore that,



So we can write:



which corresponds to the following diagrams,

A picture containing text

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which translates to:



IBP gives us:



On to the next guy,

Diagram, schematic

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This is:



which is,



Nasty. Gonna change variables q → q - q´, iqm → iqm – iqm´,



and can say,



Let’s note that:



So we can write everything in terms of G11|0(k,ikn),



And going to make some more manipulations: in 2nd term switching sign of ikn.



In 2nd term, switching (q,iqm) and (q´,iq´m),



In 2nd term switching signs of (q,iqm) and (q’,iq´m),



Alas the two terms aren’t identical. Guess I’ll write as,



where,



Let’s go straight to the semiclassical approximation whereby we assume Δ has no time-dependence, and so Δ(q,iqm) = βΔ(q)δiq\_m=0, etc. We’ll presume the same for A(q,iqm). Then we have:



and,



Well filling the G’s in,



and then recall,



So we can say,



Now it seems **Π**(2b)(0,0) = 0 since, even though the summand blows up in that limit, it is odd in **k**, and so I think the limit of the sum would be zero. Since **Π**(2b)(q,q´) is invariant under the transformation q → -q´, and q´ → -q, if there is a first order term, then it must be proportional to **Π**1(2b)(q,q´) ~ ρF\*(q-q´). So we’ll say, to lowest non-vanishing order, that:



where e\* is a renormalized charge, which we’re introducing as another free ‘tuning’ parameter. e\* will turn out to be 2e. But alas, if so, then we have,



But we could say that this is zero, if we use the Coulomb gauge. So I’ve messed up somewhere. What I really want is,



because then we’ll have:



where the first grad acts on everything to its right. And this is what we want.



Let’s skip the next two – I think RG might show these to be negligible – and just look at the last one,

Diagram, schematic

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This is:



We can make the same manipulations as on the last one, and ultimately come to:



where,



Then we’ll go straight to the semiclassical approximation whereby Δ(q,iqm) → βΔ(q)δiq\_m=0, and A2(q,iqm) → βA2(q)δiq\_m=0. Then we have:



and,



Well filling the G’s in,



and then recall,



So we can say,



Guess we’ll be approximating this by its q = q´ = 0 value, which we’ll presume to be:



Obviously this has the appropriate units, but for time’s sake, I’m assuming that the density of states prefactor will work out just so in this case. Then we’d have:



Just like with the previous term, we can write this in real space,



**O(Δ3) terms**

These will all be zero for same reason O(Δ1) terms were.

**O(Δ4) terms**

We do need to consider the Δ4 terms since when T < Tc, the O(Δ2) term changes overall sign and makes the Free Energy concave down, and so thermodynamically unstable. But turns out the only one that matters is the simplest guy,

Diagram, schematic

Description automatically generated

This is:



Working out the Trace…well FWIW, we’re eventually going to simplify to the semiclassical approximation that Δ doesn’t depend on time. This means we can make the replacement Δ(q,iqm) → βΔ(q)δiq\_m=0, and similarly with the other guys,



Then doing the trace,



Again recall,



So can say,



Now in the 2nd term going to switch q and q´, then switch sign of q´´, and then add q + q´ to q´´. Then we get:



So we can write,



where,



As usual we can do the frequency summation.



Pretty gross. But I’m going to assume (apparently true) that the lowest non-zero order term is when q = q´ = q´´ = 0. So just going to say,



Then we have:



So we have:



**Free Energy all together**

So recall we said,



and,



Now we want to put these guys in the exponent to create an effective free energy functional in terms of the order parameter Δ. So we’ll write:



where we use fact that Ξ1 = 0. And now we can write,



In the last line I took advantage of our prevailing assumption that Δ is independent of τ. So our free energy functional would be:



So let’s work this out. Filling in our results,



This reduces to:



Now let’s shift from renormalized densities of states to renormalized masses. Then we can say,



Note that we claimed ρF\* = (1/2)ρF, and so this would mean m\* = 2m. This, along with the claim that e\* = 2e, is an indication that m\*, and e\* represents the Cooper pair’s mass and charge respectively. Now let’s define,



And ψ will have units of √(E/V). Then we can say,



Now the **A** that couples to momentum operator in QM is the *total* vector potential, coming from both the external magnetic field, as well as the internal one, coming from the system’s moving charges. But if we are to have a free energy describing just the superconductor itself, the proper magnetic field variable would have to be just the external/free magnetic field, or external/free vector potential [see Thermodynamics/Equilibrium Systems]. So to accommodate the fact that we have to use the *total* vector potential, we’ll instead write down the free energy for the superconductor + magnetic field, whose proper variable *would* be the total magnetic field [see Thermodynamics/Equilibrium Systems again]. So we’ll write:



What ψ(r) physically represents, we can’t say yet, but by analogy with super-fluid thermodynamics, it was anticipated that it had something to do with the superconducting electron density. Or rather, the Cooper pair density. FWIW, we can also write this as:



where | | means magnitude in vector and complex number sense.